Optimum Inapproximability Results for Finding Minimum Hidden Guard Sets in Polygons and Terrains

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Abstract. We study the problem Minimum Hidden Guard Set, which consists of positioning a minimum number of guards in a given polygon or terrain such that no two guards see each other and such that every point in the polygon or on the terrain is visible from at least one guard. By constructing a gap-preserving reduction from Maximum 5-Ocurrence-3-Satisfiability, we show that this problem cannot be approximated by a polynomial-time algorithm with an approximation ratio of $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $NP = P$, where n is the number of polygon or terrain vertices. The result even holds for input polygons without holes. This separates the problem from other visibility problems such as guarding and hiding, where strong inapproximability results only hold for polygons with holes. Furthermore, we show that an approximation algorithm achieves a matching approximation ratio of n.

1 Introduction

In the field of visibility problems, guarding and hiding are among the most prominent and most intensely studied problems. In guarding, we are given as input a simple polygon with or without holes and we need to find a minimum number of guard positions in the polygon such that every point in the interior of the polygon is visible from at least one guard. Two points in the polygon are visible from each other, if the straight line segment connecting the two points does not intersect the exterior of the polygon. In hiding, we need to find a maximum number of points in the given input polygon such that no two points see each other.

The combination of these two classic problems has been studied in the literature as well [11]. The problem is called Minimum Hidden Guard Set and is formally defined as follows:

Definition 1. *The problem* Minimum Hidden Guard Set *consists of finding a minimum set of guard positions in the interior of a given simple polygon such that no two guards see each other and such that every point in the interior of the polygon is visible from at least one guard.*

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We can define variations of this problem by allowing input polygons to contain holes or not or by letting the input be a 2.5 dimensional terrain. A 2.5 dimensional terrain is given as a triangulated set of vertices in the plane together with a height value for each vertex. The linear interpolation inbetween the vertices defines a bivariate continuous function, thus the name 2.5 dimensional terrain (see [10]). In other variations, the guards are restricted to sit on vertices. Problems of this type arise in a variety of applications, most notably in telecommunications, where guards correspond to antennas in a network with a simple line-of-sight wave propagation model (see [4]).

While Minimum Hidden Guard Set is NP-hard for input polygons with or without holes [11], no approximation algorithms or inapproximability results are known. For other visibility problems, such as guarding and hiding, the situation is different: Minimum Vertex/Point/Edge Guard are NP-hard [9] and cannot be approximated with an approximation ratio that is better than logarithmic in the number of polygon or terrain vertices for input polygons with holes or terrains [4]; these problems are APX-hard[1] for input polygons without holes [4]. The best approximation algorithms for these guarding problems achieve a logarithmic approximation ratio for Minimum Vertex/Edge Guard for polygons [8] and terrains [6], which matches the logarithmic inapproximability result upto low-order terms in the case of input polygons with holes and terrains; the best approximation ratio for Minimum Point Guard is $\Theta(n)$, where n is the number of polygon or terrain vertices. The problem Maximum Hidden Set cannot be approximated with an approximation ratio of n^ϵ for some $\epsilon > 0$ for input polygons with holes and it is APX-hard for polygons without holes ([5] or [7]). The best approximation algorithms achieve approximation ratios of $\Theta(n)$. Thus, for both, hiding and guarding, the exact inapproximability threshold is still open for input polygons without holes. To get an overview of the multitude of results in visibility problems, consult [12] or [13].

In this paper, we present the first inapproximability result for Minimum Hidden Guard Set: we show that no polynomial-time algorithm can guarantee an approximation ratio of $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $NP = P$, where n is the number of vertices of the input structure. The result holds for terrains, polygons with holes, and even polygons without holes as input structures. We obtain our result by constructing a gap-preserving reduction (see [1] for an introduction to this concept) from Maximum 5-Occurrence-3-Satisfiability, which is the APX-hard satisfiability variation, where each clause consists of at most three literals and each variable occurs at most five times as a literal [2]. We also analyze an approximation algorithm for Minimum Hidden Guard Set proposed in [11] and show that it achieves a matching approximation ratio of n.

[1] A problem is in the class APX, if it can be approximated by a polynomial-time algorithm with an approximation ratio of $1 + \delta$, for some constant $\delta \geq 0$. It is APX-hard, if no polynomial-time algorithm can guarantee an approximation ratio of $1 + \epsilon$, for some constant $\epsilon > 0$, unless $P = NP$. A problem is APX-complete, if it is in APX and APX-hard. See [2] for more details.

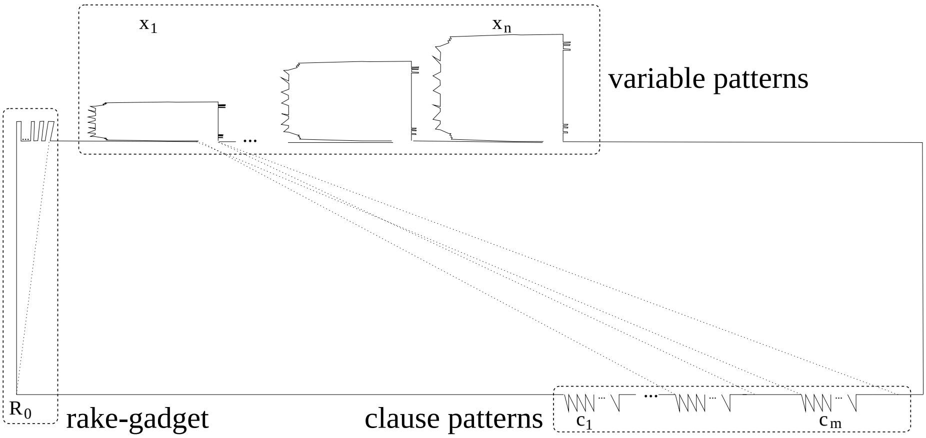

Fig. 1. Overview of construction

In Sect. 2 we present the construction of the reduction. We analyze the reduction and obtain our main result in Sect. 3. We analyze an approximation algorithm in Sect. 4. Section 5 contains some extensions of our results and concluding thoughts.

2 Construction of the Reduction

In this section, we show how to construct in polynomial time from a given instance I of Maximum 5-Occurrence-3-Satisfiability with n variables $x_1, \ldots, x_n$ and m clauses $c_1, \ldots, c_m$ an instance I' of Minimum Hidden Guard Set, i.e., a simple polygon.

An overview of the construction is given in Fig. 1. The main body of the constructed polygon is of rectangular shape. For each clause c_i, a *clause pattern* is constructed on the lower horizontal line of the rectangle, and for each variable x_i, we construct a *variable pattern* on the upper horizontal line as indicated in Fig. 1.

The construction will be such that a variable assignment that satisfies all clauses of I exists, if and only if the corresponding polygon I' has a hidden guard set with $O(n)$ guards; otherwise, I' has a hidden guard set of size $O(t)$, where t will be defined as part of the *rake-gadget* in the construction. The rake gadget, shown in Fig. 2, enables us to force a guard to a specific point R in the polygon. It consists of t dents, which are small trapezoidal elements that point towards point R. Rakes have the following property:

Lemma 1. *If the t dents of a rake are not covered by a single hidden guard at point R, then t hidden guards (namely one guard for each dent) are necessary to cover the dents.*

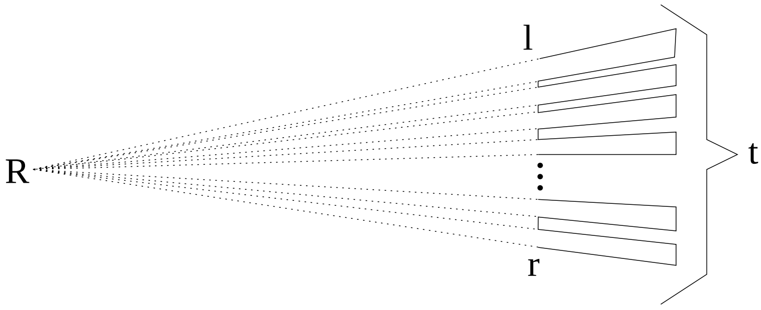

Fig. 2. Rake with t dents

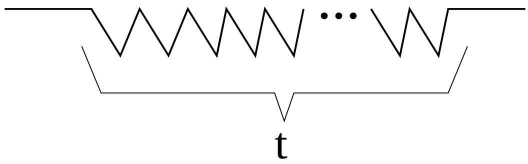

Fig. 3. Clause pattern consisting of t triangles

Proof. Clearly, any guard outside the triangle R, l, and r and outside the dents does not see a single dent completely. A guard in this triangle (but not at R) sees at most one dent completely, but only one such guards can exist as guards must be hidden from each other. Therefore, at least $t-1$ guards must be hidden in the dents. □

In order to benefit from this property of a rake, we must place the rake in the polygon in such a way that the view from point R to the rake dents is not blocked by other polygon edges. As shown in Fig. 1, we place a rake at point R_0 in the lower left corner of the rectangle with the t dents at the top left corner.

A clause pattern, shown in Fig. 3, consists of t triangular-shaped spikes. Clause patterns are placed on the lower horizontal line of the rectangle. They are constructed in such a way that a guard on the upper horizontal line could see all spikes of all clause patterns. (This, however, will never happen, as we have already forced a guard to point R_0 to cover its rake. This guard would see any guard on the upper horizontal line.)

For each variable x_i, we construct a variable pattern, that is placed on top of the horizontal line of the rectangle. Each variable pattern opens the horizontal line for a unit distance. Each variable pattern has constant distance from its neighbors and the right-most variable pattern (for variable x_n) is still to the left of the left-most clause pattern (for clause c_1), as indicated in Fig. 1. The variable patterns will differ in height, with the left-most variable pattern (for x_1) being the smallest and the right-most (for x_n) the tallest. Figure 4 shows the variable pattern of variable x_i.

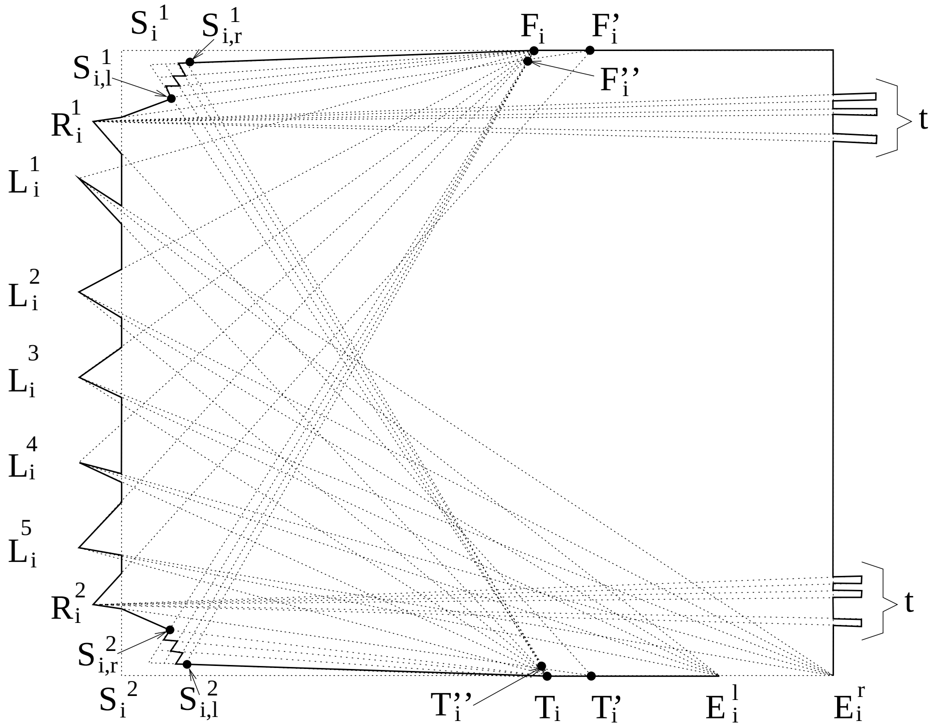

Fig. 4. Variable pattern with three positive and two negative literals

A variable pattern is roughly a rectangular structure with a point F_i on top and a point T_i on the bottom. The construction is such that a guard sits at F_i, if the variable is set to false, and at T_i otherwise. Literals are represented by triangles with tips $L_i^1, \ldots, L_i^5$ for each of the five occurrences of the variable (some may be missing, if a variable occurs less than five times as literal). These triangles are constructed such that – for positive literals – they are completely visible from F_i, but not from T_i, and – for negative literals – they are completely visible from T_i, but not from F_i. A guard that sits at a point L_i^k, for any $k = 1, \ldots, 5$, can see through the exit of the variable pattern between points E_i^l and E_i^r. The construction is such that such a guard sees all spikes of the corresponding clause pattern (but no spikes of other clause patterns). This is shown schematically in Fig. 1.

In order to force a guard to sit at either F_i or T_i, we construct a rake point R_i^1 above L_i^1 and a rake point R_i^2 below L_i^5 with t dents, all of which are on the right vertical line of the variable rectangle. Points R_i^1 and R_i^2 are at the tip of small triangles that point towards points F_i' and T_i', which lie a small distance to the right of F_i and T_i, respectively. In addition, we construct two areas S_i^1 and S_i^2 to the left of T_i and F_i, where we put t triangular spikes, each pointing exactly towards F_i and T_i. For simplicity, we have only drawn three triangular spikes in Fig. 4 instead of t. Area S_i^1 is the area of all these triangles at the top of the variable rectangle, area S_i^2 is the area of all these triangles at the bottom of the variable rectangle.

This completes our description of the constructed polygon that is an instance of Minimum Hidden Guard Set. The polygon consists of a number of vertices that is polyonmial in the size $|I|$ of the Max 5-Occurrence-3-Satisfiability instance I and in t. The coordinates of each vertex can be computed in time polynomial in $|I|$ and t, and they can be expressed by a polynomial (in $|I|$ and t) number of bits. Thus, the reduction is polynomial, if t is polynomial in $|I|$.

3 Analysis of the Reduction

The following two lemmas describe the reduction as gap-preserving and will allow us to prove our inapproximability result.

Lemma 2. *If the* Maximum 5-Occurrence-3-Satisfiability *instance I with n variables can be satisfied by a variable assignment, then the corresponding* Minimum Hidden Guard Set *instance I' has a solution with at most $8n + 1$ guards.*

Proof. In I', we set a guard at each rake point R_0 and R_i^1 and R_i^2, for $i = 1, \ldots, n$, which gives a total of $2n+1$ hidden guards. For each variable x_i, we then place a guard at F_i or T_i depending on the truth value of the variable in a fixed satisfying truth assignment; this yields additional n hidden guards. Finally, we place a guard at each literal L_i^k, if and only if the corresponding literal is true. This yields at most $5n$ hidden guards, as each variable occurs at most five times as a literal. Since the thruth assignment satisfies all clauses, all clause patterns will be covered by at least one guard. The variable patterns and the main body rectangle are covered completely as well. Thus, the solution is feasible and consists of at most $8n + 1$ guards. □

Lemma 3. *If the* Maximum 5-Occurrence-3-Satisfiability *instance I with n variables cannot be satisfied by a variable assignment, then any solution of the corresponding* Minimum Hidden Guard Set *instance I' has at least t guards.*

Proof. We prove the following equivalent formulation: If I' has a solution with strictly less than t guards, then I is satisfiable.

Assume we have a solution for I' with less than t guards. Then, there must be a guard at each rake point R_0 and R_i^1 and R_i^2 for $i = 1, \ldots, n$; this already restricts the possible positions for all other guards quite drastically, since they must be hidden from each other.

Observe in this solution, how the triangles of the areas S_i^1 and S_i^2 are covered. Since we have guards at rake points R_i^1 and R_i^2, the guards for S_i^1 and S_i^2 can only lie in the 4-gons $(S_{i,l}^1, S_{i,r}^1, F_i, F_i')$ or $(S_{i,r}^2, S_{i,l}^2, T_i, T_i')$, but only a guard in the smaller triangle of either (F_i, F_i', F_i'') or (T_i, T_i', T_i'') can see both areas S_i^1

and S_i^2 (see Fig. 4). If S_i^1 or S_i^2 is covered by a guard outside these triangles, then the other area can only be covered with t guards inside the S_i^1 or S_i^2 triangles. Therefore, there must be a guard in either one of the two triangles (F_i, F_i', F_i'') or (T_i, T_i', T_i'') in each variable pattern. (Point F_i'' is the intersection point of the line from R_i^1 to F_i' and from $S_{i,r}^2$ to F_i; Point T_i'' is the intersection point of the line from R_i^2 to T_i' and from $S_{i,l}^1$ to T_i). We can move this guard to point F_i or T_i, respectively, without changing which literal triangles it sees.

Now, the only areas in the construction not yet covered are the literal triangles of those literals that are true and the spikes of the clause patterns. Assume for the sake of contradiction that one guard is hidden in a triangle of a clause pattern c_i. This guard sees the triangles of all literals that represent literals from the clause. This, however, implies that the remaining $t-1$ triangles of the clause pattern c_i can only be covered by $t-1$ additional guards in the clause pattern, thus resulting in t guards total. Therefore, all remaining guards must sit in the literal triangles in the variable patterns. W.l.o.g., we assume that there is a guard at each literal point L_i^k that is not yet covered by a guard at points F_i or T_i. If these guards collectively cover all clause patterns, we have a satisfying truth assignment; if they do not, at least t guards are needed to cover the remaining clause patterns. □

Lemmas 2 and 3 immediately imply that we cannot approximate MINIMUM HIDDEN GUARD SET with an approximation ratio of $\frac{t}{8n+1}$ in polynomial time, because such an algorithm could be used to decide MAXIMUM 5-OCCURRENCE-3-SATISFIABILITY. To get to an inapproximability result, we first observe that

$$|I'| \leq (8t+30)n + 2tm + 4t + 100 \leq 18tn + 30n + 4t + 100$$

by generously counting the constructed polygon vertices and using $m \leq 5n$. We now set

$$t = n^k$$

for an arbitrary but fixed $k > 1$. This implies $|I'| \leq n^{k+2}$ and thus

$$n \geq |I'|^{\frac{1}{k+2}}$$

On the other hand, we cannot approximate MINIMUM HIDDEN GUARD SET with an approximation ratio of

$$\frac{t}{8n+1} \geq \frac{n^k}{n^2} = n^{k-2} \geq |I'|^{\frac{k-2}{k+2}} = |I'|^{1-\frac{4}{k+2}}.$$

Since k is an arbitrarily large constant, we have shown our main theorem:

Theorem 1. MINIMUM HIDDEN GUARD SET *on input polygons with or without holes cannot be approximated by any polynomial time approximation algorithm with an approximation ratio of* $|I|^{1-\epsilon}$ *for any* $\epsilon > 0$*, where* $|I|$ *is the number of polygon vertices, unless* $NP = P$.

4 An Approximation Algorithm

The following algorithm to find a feasible solution for Minimum Hidden Guard Set was proposed in [11]: Iteratively add a guard to the solution by placing it in an area of the input polygon (or terrain) that is not yet covered by any other guard that is already in the solution. In terms of an approximation ratio for this algorithm, we have the following

Theorem 2. Minimum Hidden Guard Set *can be approximated in polynomial time with an approximation ratio of* $|I|$*, where* $|I|$ *is the number of polygon vertices.*

Proof. Any triangulation of the input polygon partitions the polygon into $|I|-2$ triangles. Now, fix any triangulation. Any guard that the approximation algorithm places (as described above) lies in at least one of the triangles of the triangulation and thus sees the corresponding triangle completely. Therefore, the solution will contain at most $|I|-2$ guards. Since any solution must consist of at least one guard, the result follows. □

5 Extensions and Conclusion

Theorem 1 extends straight-forward to terrains as input structures by using the following transformation from a polygon to a terrain (see [4]): Given a simple polygon, draw a bounding box around the polygon and then let all the area in the exterior of the polygon have height h (for some $h > 0$) and the interior height zero. This results in a terrain with vertical walls that we then triangulate. Similarly, Theorem 2 extends to terrains as input structures immediately.

Another straight-forward extension of Theorem 1 leads to problem variations, where the guards may only sit at vertices of the input structure. Since we have always placed or moved guards to vertices throughout our construction, Theorem 1 holds for Minimum Hidden Vertex Guard Set for input polygons with or without holes and terrains. Unfortunately, the vertex-restricted problem variations cannot be approximated analogously to Theorem 2, as even the problem of determining whether a feasible solution exists for these problems is NP-hard [11].

If we restrict the problem even more, namely to a variation, where the guards may only sit at vertices and they only need to cover the vertices rather than the whole polygon interior, we arrive at the problem Minimum Independent Dominating Set for visibility graphs. Also in this case, Theorem 1 holds, thus adding the class of visibility graphs to the numerous graph classes for which this problem cannot be approximated with a ratio of $n^{1-\epsilon}$. The approximation algorithm from Sect. 3 can be applied for this variation and achieves a matching ratio of n.

The complementary problem Maximum Hidden Guard Set, where we need to find a maximum number of hidden guards that cover a given polygon, is equivalent to Maximum Hidden Set. Therefore, it cannot be approximated with an approximation ratio of n^ϵ for some $\epsilon > 0$ for input polygons with holes and it is APX-hard for input polygons without holes [7]. The corresponding vertex-restricted variation cannot be approximated, as it is – again – NP-hard to even find a feasible solution.

We have presented a number of inapproximability and approximability results for Minimum Hidden Guard Set in several variations. Most results are tight upto low-order terms. However, there still exists a large gap regarding the inapproximability of the problem Maximum Hidden Guard Set on input polygons without holes, where only APX-hardness is known and the best approximation algorithms achieve approximation ratios of $\Theta(n)$.

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